Homework 4 Solutions

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1 Asymptotic Normality to Consistency

A) Consistency is a basic requirement of an estimator. As long as our estimator is consistent, an arbitrarily large sample will allow us to learn the true population value of the parameter. If we don't have consistency then it doesn't matter how much data we have, we may never learn the value of the population parameter. As the *finite* sample distributions of estimators are rarely known, we rely on *asymptotic* distribution to inform us of the critical values for test statistics. If we know that our estimator is asymptotically normal, we can use the critical values for the normal distribution to form tests of statistical significance.

B) You begin by giving the geologist a pat on the back (or maybe a chest-bump if it is a more formal setting). The asymptotic normality result does in fact establish consistency. Note that:

$$\frac{1}{\sqrt{n}}\sqrt{n}(\hat{\theta}_n - \theta_0) = \hat{\theta}_n - \theta_0.$$

$\sqrt{n}(\hat{\theta}_n - \theta_0)$ converges in distribution to a random variable and $\frac{1}{\sqrt{n}}$ converges in in probability to zero. The product of the two, therefore, converges in probability to zero. This implies that $\hat{\theta}_n - \theta_0$ converges in probability to zero or, equivalently, that $\hat{\theta}_n$ converges in probability to θ_0 and thus we have a consistent estimator.

C) The rate of convergence of the new estimator is much greater than that of the estimator discussed above. The estimator that converges at n has an equivalent variance with 10 observations as the other estimator (which converges at $\sqrt{n}$) has with 100 observations. The new estimator uses available information much more efficiently than the estimator mentioned in (B) and would be preferred.

2 Asymptotic Convergence

A) An individual's payoffs in the lottery are:

$$p_n = \begin{cases} 0 & \text{with probability } \frac{n-1}{n} \\ n^2 & \text{with probability } \frac{1}{n} \end{cases}$$

A risk neutral individual cares only about the expected winnings, and so will play only if expected winnings are positive:

$$E(w_n) = E(p_n) - \frac{n}{2} = 0 \cdot \frac{n-1}{n} + n^2 \cdot \frac{1}{n} - \frac{n}{2} = \frac{n}{2}$$

As expected winnings are positive, the risk neutral individual will be visiting his/her local 7Eleven to purchase a ticket.[1]

B) To assess convergence in probability, we are interested in computing $\Pr(|p_n| > \epsilon)$ for any $\epsilon > 0$. In this case, we actually know the probability with which p_n will yield non-zero returns.

$$\Pr(|p_n| > \epsilon) \leq \frac{1}{n}$$

Because $\lim_{n\to\infty} \frac{1}{n} = 0$ we can show:

$$\lim_{n\to\infty} \Pr(|p_n| > \epsilon) = 0$$

Thus the individual's payoff converges to zero in probability.

C) The individual's expected payoff is $E(p_n) = 0 \cdot \frac{n-1}{n} + n^2 \cdot \frac{1}{n}$ which is simply equal to n. As the number of tickets sold increases, we are concerned with $E(p_n)$ as n increases: $\lim_{n\to\infty} E(p_n) = \infty$. The individual's expected payoff grows without bound as n increases – the expected payoff diverges.

D) No. While it is the case that for large values of n individuals receive no payoff with probability approaching one, the size of the payoff (if they do win) grows in such a way that the expected value of buying a ticket grows. Thus risk-neutral individuals would always play the lottery.

3 G and γ

A) To obtain the asymptotic distribution for G we employ the delta method (recall 241A). Let $\gamma = g(\beta)$ so $g(\beta) = \frac{1}{\beta}$. The first derivative of $g(\beta)$ is $\frac{-1}{\beta^2}$, which is continuous at $\beta \neq 0$. By the delta method

$$\sqrt{n}(G - \gamma) \to_d N(0, \frac{\sigma^2}{\beta^4})$$

Thus G is a consistent estimator for (converges in probability to) γ by the same logic as in question 1:

$$\frac{1}{\sqrt{n}}\sqrt{n}(G - \gamma) = G - \gamma.$$

Note that you could have arrived the conclusion that G is a consistent estimator for γ by noting that B is consistent for β and then employing the Continuous Mapping Theorem.

B) The estimated value of g is simply $\frac{1}{6}$. To obtain the estimated asymptotic standard errors, note that

[1] Most lotto ticket revenues go to support schools. Also, most lotto tickets are purchased by low income household. Fun facts.

$$(B - \beta) \sim_A N(0, \frac{\sigma^2}{n})$$

Our estimated asymptotic standard errors for B equal 4 ($= \sqrt{\frac{320}{20}}$) and for G equal $\frac{1}{9}$ ($= \sqrt{\frac{\sigma^2}{n}\frac{1}{\beta^4}} = \sqrt{16\frac{1}{6^4}}$).

C) The asymptotic two-sided confidence interval for B is $(6 \pm 1.96 \cdot 4)$, where 1.96 is the appropriate critical value. Because $2 \in (6 \pm 1.96 \cdot 4)$, we cannot reject the null.[2]

D) The corresponding hypotheses for γ are $H_0 : \gamma = \frac{1}{2}$ against $H_A : \gamma \neq \frac{1}{2}$. The approximate two-sided confidence interval for G is $(\frac{1}{6} \pm 1.96 \cdot \frac{1}{9}) = (-0.051, 0.384)$. The upper bound of this interval is less than $\frac{1}{2}$. We can reject the null with G even though we could not with B, so we have a more precise estimator of γ than we do for β if $\beta > 1$.

An important take away is the logic of the continuous mapping theorem does *not* apply directly to confidence intervals – significance is not robust to (non-linear) transformations. In general, tests derived from the Wald test are not robust to transformations of the underlying hypothesis. This is in contrast to some other tests (i.e. the likelihood ratio test), despite their asymptotic equivalence.

[2] Note that we are applying asymptotic theory with a sample size of twenty ($n = 20$). This may be problematic – if normal errors could be assumed, critical values drawn from the t-distribution would be more appropriate. Without the normality assumption...